Cross Sections

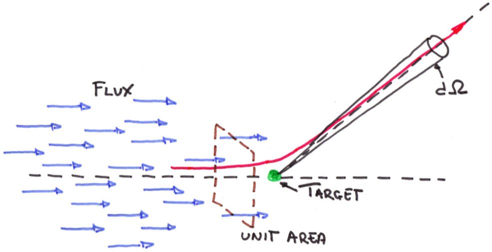
Radiation-Matter Interaction

Th. Cocolios, N. Severijns (KU Leuven)
E.Cortina (UCLouvain)

Cross Section (σ)

- Collision or interaction between two particles is described in terms of cross section
- It gives a measure of the probability for a reaction to occur
- It can be calculated on the knowledge of the fundamental interaction between particles
- We define the differential cross section as the probability to observe a scattered particle per solid angle unit.

- It depends on:
  - Particles
  - Type of interaction
  - Energy
  - Angle



Differential cross section

- Let's consider a beam of incident particle of type A
- And a target formed by ONE particle of type B
- The beam is broader than the target and the particles are uniformly distributed in time and space
- We define the flux as

$$\text{Flux} = \Phi = \frac{\#\ \text{incident particles}}{(\text{Unit Area})\ (\text{Unit time})}$$

- Particles not interacting = Transmitted particles
  Particles interacting = Scattered particles (N_s)

dN_s	=	P	$\cdot$	Φ	$\cdot$	$d\Omega$
part/s		Probability Interaction Units=cm^2		part/(cm^2 s)		Geometrical factor (dimensionless)

Differential cross section

- Probability of interaction is given by the differential cross section

$$dN_s = P \cdot \Phi \cdot d\Omega$$

$$\frac{d\sigma}{d\Omega}(E,\Omega) = \frac{1}{\Phi}\frac{dN_s}{d\Omega}$$

  - dimensions = area
  - heuristic interpretation: geometrical cross-section of the target intercepting the beam.
    Please do not mix concepts!!!
  - Units : 1 barn $= 10^{-24}$ cm^2
    $= 10^{-28}$ m^2

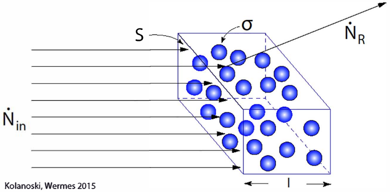
- We define the total cross section as

$$\sigma(E) = \int d\Omega \frac{d\sigma}{d\Omega}$$

Measurement of the cross section

- Let's consider a thin slab of material: area S and length ℓ
  - thin = Just 1 interaction per incident particle
    No auto-absorption
- Let's consider that the beam surface is equal to the target one $S = S_b$
- Experimentally we can measure the cross section by counting the particles scattered in a target from a flux Φ of incoming particles:

$$P_{int} = \frac{N_R}{N_{in}} \quad \Rightarrow \begin{cases} N_{in}\text{: Number (or rate) of incoming particles} = \Phi S \\ N_R\text{: Number (or rate) of scattered particles} \end{cases}$$



Measurement of the cross section

- From the cross section definition:
  - effective area "seen" by an incoming particle $S_{eff} = N_T\sigma$

$$N_T = \frac{\rho V}{A} N_A \quad \Rightarrow \begin{cases} N_T: & \text{Number of targets in the volume} \\ N_A: & \text{Avogadro's number} \\ A: & \text{mass per mole} \end{cases}$$

  - The total area of the target is therefore the ratio between "effective" and geometrical surface:

$$P_{int} = \frac{S_{eff}}{S} = \frac{N_T\sigma}{S} = N\sigma\ell$$

where N is the so called particle number density:

$$N = \frac{N_T}{V} = \frac{\rho}{A} N_A$$

- Anagously we define the electron number density as:

$$N_e = ZN = \frac{Z\rho}{A} N_A$$

Measurement of the cross section

- The number of scattering centers per unit perpendicular area to the beam is: $N\ell$
- The total number of scattering centers is $NS\ell$

$$dN_R = \frac{d\sigma}{d\Omega}\Phi d\Omega \qquad \text{for 1 scattering center}$$

$$dN_R = \frac{d\sigma}{d\Omega}NS\ell\Phi d\Omega \qquad \text{for } NS\ell \text{ scattering centers}$$

$$N_R = \Phi SN\ell\sigma = N_{in}N\ell\sigma$$

- Combining all the previous equations we obtain:

$$P_{int} = \frac{N_R}{N_{in}} = N\sigma\ell \quad \Rightarrow \quad \sigma = \frac{1}{N\ell}\frac{N_R}{N_{in}}$$

- Number of collisions per unit length of a particle is: $N\sigma$

Measurement of Cross Section

- If the target surface (S_t) is smaller than the beam surface (S_b):
  - The same expression is still valid
  - Now the area to be considered is that of the beam S_t
  - $S_t\Phi =$ number of incident particles

$$N_R = \Phi S_t N \delta_x \sigma \quad \Rightarrow \quad \sigma = \frac{1}{N\ell}\frac{N_R}{\Phi S_t}$$

- In case that $S_b < S_t$ the surface to be consider is S_b:

$$N_R = \Phi S_b N \delta_x \sigma \quad \Rightarrow \quad \sigma = \frac{1}{N\ell}\frac{N_R}{\Phi S_b}$$

Survival Probability

- What's the probability on not having an interaction in a distance x?

$P(x)$	$=$	probability of not having an interaction in a distance x
ω	$=$	probability of having an interaction per unit length
ωdx	$=$	probability of having an interaction between x and $x+dx$

Probability of not having an interaction between x and $x+dx$ is

$$\left.\begin{aligned} P(x+dx) &= P(x)(1-\omega dx) \\ P(x)+\frac{dP(x)}{dx}dx &= P(x)-\omega P(x)dx \\ dP(x) &= -P(x)\omega dx \end{aligned}\right\} \rightarrow \begin{aligned} &P(x) = Ce^{-\omega x} \\ &P(0) = 1 \\ &P(x) = e^{-\omega x} \end{aligned}$$

- $P(x)$= survival probability
- The probability of suffering an interaction anywhere in the distance x is

$$P_{int} = 1 - P(x)$$

Interaction probability

- The probability of interaction between x and $x + dx$ is:

$$P_{int}(x)dx = \text{Survival Probability} \times \text{Interaction probabilty in } dx$$
$$P_{int}(x)dx = e^{-\omega x} \times \omega dx$$

- We can then calculate the mean free path as the mean distance in which there is not an interaction :

$$\lambda = \frac{\int xP(x)dx}{\int P(x)dx} = \frac{\int xe^{-\omega x}dx}{\int e^{\omega x}dx} = \frac{1}{\omega}$$

Mean Free Path

- λ must be related with the density of interaction centers and the cross-section.
  For a small δx we can write the interaction probability as:

$$P_{int}(\delta x) = 1 - P(\delta x) \simeq 1 - \left(1 - \frac{\delta x}{\lambda} + \cdots\right) \simeq \frac{\delta x}{\lambda}$$

- This quantity has been already calculated:

$$P_{int}(\delta x) = N\sigma\delta x$$

- From the two expression we get

$$\lambda = \frac{1}{N\sigma} = \frac{A}{\rho N_A \sigma}$$

Energy loss of charged particles

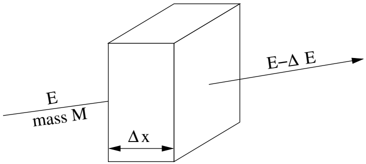
- Interaction of the radiation with the atomic electrons is the dominant energy loss process for charged particles.
  - First calculated classically by Bohr (1913)
  - Bethe and Bloch provided quantum mechanical calculation (1930's)
  - Subsequently refined: correction for various kinematical regimes.
- The average energy loss of a particle with mass M and velocity β can be expressed as:

$$-\left\langle \frac{dE}{dx} \right\rangle = n_e \int_{T_{min}}^{T_{max}} T \frac{d\sigma}{dT}(M, \beta, T) dT$$

T: Kinetic energy loss in the collision

$n_e = \frac{Z\rho}{A} N_A$: Target number density

$\sigma(M,\beta,T)$: Cross section



Rutherford scattering

$$-\left\langle \frac{dE}{dx} \right\rangle = n_e \int_{T_{min}}^{T_{max}} T \frac{d\sigma}{dT} dT$$
$$= n_e \int_{T_1}^{T_{max}} T \frac{d\sigma}{dT} dT - \left\langle \frac{dE}{dx} \right\rangle_{T<T_1}$$

- In the following we are going to assume:
  - Incoming particle is "heavy": $M >> m_e$
  - For $T > T_1$: "quasi-free" orbital electrons ($\beta >> \beta_e$)
  - T_1 larger than ionization energy: $T_1 \approx 0.01 - 0.1$ MeV
- Let's consider incoming and outgoing 4-vectors: P, p_e, P', p'_e.
- 4-momentum transfer: $Q^2 = -(P - P')^2 = -(p_e - p'_e)^2 = 2 m_e c^2 T$
- Lorentz-invariant Rutherford cross section:

$$\frac{d\sigma}{dQ^2} = \frac{4\pi z^2 \alpha^2 \hbar^2 c^2}{\beta^2} \frac{1}{Q^4} \Longrightarrow \frac{d\sigma}{dT} = \frac{2\pi z^2 \alpha^2 \hbar^2}{\beta^2 m_e} \frac{1}{T^2}$$

Rutherford scattering

- Mott cross section takes into account possible spin flip of the target electron

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 \alpha^2 \hbar^2}{\beta^2 m_e T^2}\left(1 - \beta^2 \frac{T}{T_{max}}\right)$$

- We can now solve the integral

$$\left\langle \frac{dE}{dx} \right\rangle_{T>T_1} = n_e \int_{T_1}^{T_{max}} T \frac{2\pi z^2 \alpha^2 \hbar^2}{\beta^2 m_e T^2}\left(1 - \beta^2 \frac{T}{T_{max}}\right) dT$$
$$= \frac{2\pi z^2 \alpha^2 \hbar^2}{\beta^2 m_e} n_e \left(\ln \frac{T_{max}}{T_1} - \beta^2\right)$$

- It contains most of the dependencies of energy loss.

Minimum Energy Transfer

- Classically the energy transferred to a free electron can become arbitrarily small.
- Quantum mechanically:
  - Below the ionisation threshold: only discrete energy transfers.
  - If particle velocities similar or smaller than electron orbitals: interference effects plays a role.
  - We have to take into account excitations, atomic screening,....
- Rigorous treatment by Bethe:

$$\left\langle \frac{dE}{dx} \right\rangle_{T<T_1} = \frac{2\pi z^2 \alpha^2 \hbar^2}{\beta^2 m_e} n_e \left(\ln \frac{2 m_e c^2 \beta^2 T_1}{I^2} - \ln \frac{1}{\gamma^2} - \beta^2 \right)$$

  - Integral between an effective $T_{min} = \frac{I^2}{2 m_e c^2 \beta^2}$ and $T_{max} = T_1$
  - I is the mean excitation energy (0.1 eV - few eV).
  - The term $\ln \frac{1}{\gamma^2}$ accounts for a relativistic growth at high energies.

Cross Sections

- The scattering of heavy particles with charge ze with free electrons is described by the Rutherford differential cross section

$$\frac{d\sigma_R(E,\beta)}{dE} = \frac{2\pi r_e^2 m_e c^2 z^2}{\beta^2} \frac{(1-\beta^2 E/T_{max})}{E^2}$$

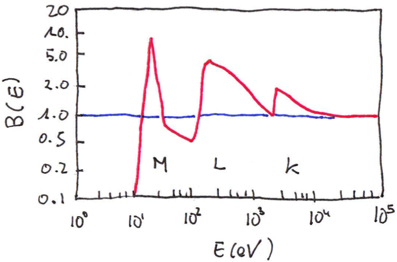
where

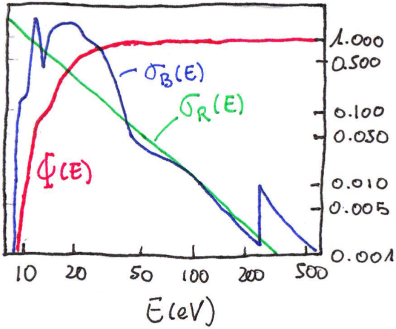
r_e : Classical electron radius
m_e : Electron mass
z : Charge of the incident particle
β : Velocity of the incident particle
E : Energy loss of the incident particle
T_{max} : Maximum energy transfer possible in a single collision

- Electrons in matter are not free. Bethe took into account this with the inclusion of a correction factor

$$\frac{d\sigma_B(E,\beta)}{dE} = \frac{d\sigma_R(E,\beta)}{dE} B(E)$$

Cross Sections





Moments

- Cross-Section = Probability of interaction
- Mean number of collisions with energy loss between E and $E+dE$ in a distance δx is:

$$N = N_e \delta x \left(\frac{d\sigma(E,\beta)}{dE} \right) dE$$

where

N_e : Electron density = ZN

- It's convenient to define the moments

$$M_j(\beta) = N_e \delta x \int E^j \frac{d\sigma(E,\beta)}{dE} dE$$

M_0 : mean number of collisions in δx

M_1 : mean energy loss in δx

$M_2 - M_1^2$: Variance

  - Usually defined per unit length

- The number of collisions is Poisson distributed with mean $m_c = M_0 x$

Macroscopic Cross Section

- We define the macroscopic cross section as

$$\Sigma_t(\gamma\beta) = N_e \int \frac{d\sigma(E,\beta)}{dE} dE = M_0 \qquad m_c = x\Sigma_t(\gamma\beta)$$

- Previously we calculated the mean number of collision per unit length as

$$\frac{1}{\lambda} = N_e \sigma(\gamma\beta) \qquad \lambda = \frac{1}{\Sigma_t}$$